AS-2820

B.Sc. (Hon's) (First Semester) Examination, 2013

Mathematics

Paper : Second

(Calculus)

Model Answer / Suggestive alternate Solution.

[1] Answer the following questions:

(i) If $y = e^{\sin x^3}$ find $\frac{d^3y}{dx^3}$.

Ans

$$\frac{dy}{dx} = e^{\sin x^3} . \cos x^3 . 3x^2$$

Again differentiating w.r.to x,

$$\frac{d^2y}{dx^2} = e^{\sin x^3} . (\cos x^3 . 3x^2)^2 + e^{\sin x^3} . (-\sin x^3 . 3x^2) . 3x^2 + e^{\sin x^3} . \cos x^3 . 6x .$$

$$= e^{\sin x^3} \{ 9x^4 \cos^2 x^3 - 9x^4 \sin x^3 + 6x \cos x^3 \}$$

Again diff. w.r.to x,

$$\frac{d^3y}{dx^3} = e^{\sin x^3} . \cos x^3 . 3x^2 \{ 9x^4 \cos^2 x^3 - 9x^4 \sin x^3 + 6x \cos x^3 \}$$
$$+ e^{\sin x^3} \{ 36x^3 \cos^2 x^3 + 18x^4 \cos x^3 . (-\sin x^3) . 3x^2 - 36x^3 \sin x^3 - 9x^4 \cos x^3 . 3x^2 + 6 \cos x^3 - 6x \sin x^3 . 3x^2 \} .$$

(ii) Write expression for $D^n(e^{ax} \sin(bx+c))$

Ans $D^n(e^{ax}\sin(bx+c)) = r^n e^{ax} \sin(bx + c + n\phi)$,

where, $r = (a^2+b^2)^{1/2}$ and $\phi = \tan^{-1}(b/a)$.

(iii) Write chain rule of differentiation.

Ans The chain rule expresses the derivative of the composite function $f \circ g$ in terms of the derivative of f and g. i.e.

$$\frac{d(f \circ g)}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

where f is a function of g and g is a function of x.

(iv) Compute $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

Ans:-

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = 1.$$

(v) Verify Rolle's Theorem for $f(x) = x^3 - 4x$ in $[-2, 2]$.

Ans:- As the given function is a polynomial, it is continuous in the closed interval $[-2, 2]$ and differentiable in $(-2, 2)$.

Again $f(-2) = (-2)^3 - 4(-2) = -8 + 8 = 0$

$f(2) = (2)^3 - 4 \cdot (2) = 8 - 8 = 0.$

So $f(-2) = f(2)$.

So we must have a point $c \in (-2, 2)$ s.t $f'(c) = 0$ where, $f'(x) = 3x^2 - 4$

i.e $3c^2 - 4 = 0$

$\Rightarrow c^2 = 4/3$

$\Rightarrow c = \frac{2}{\sqrt{3}}$

(vi) leave it.

(vii) Reduce the following into definite integral:

$$\lim_{n \to \infty} \sum_{r=0}^{\infty} \frac{1}{n} f\left(\frac{r}{n}\right).$$

Ans:- $\lim_{n\to\infty} \sum_{r=0}^{\infty} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$

(viii) Find asymptotes for the curve $xy = 4$.

Ans: Asymptote parallel to x axis is $y = 0$.

and Asymptote parallel to y-axis is $x = 0$;

As the degree of the given curve is two. So it can not have more than two asymptotes.

2. (a) If $y = \tan^{-1}\frac{x}{a}$, find y_n.

Ans If $y = \tan^{-1}\frac{x}{a}$, then $y_1 = \frac{a}{a^2+x^2}$

Now, $\frac{1}{a^2+x^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ai}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$

Therefore $y_n = \frac{a(-1)^{n-1}(n-1)!}{2ia}\left\{\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right\}$.

For more simplification $\left[\because D^n(ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}\right.$

put $x = r\cos\phi$, $a = r\sin\phi$, then

$y_n = \frac{1}{2}(-1)^n(n-1)!\; i\; r^{-n}\left\{(\cos\phi - i\sin\phi)^{-n} - (\cos\phi + i\sin\phi)^{-n}\right\}$

$= \frac{1}{2}(-1)^n(n-1)!\; i\; r^{-n}\left\{\cos n\phi + i\sin n\phi - \cos n\phi + i\sin n\phi\right\}$

$= (-1)^{n+1}(n-1)!\; r^{-n}\sin n\phi$ where $r^{-n} = a^{-n}\sin^n\phi$ & $\phi = \tan^{-1}(a/x)$

$\Rightarrow y_n = (-1)^{n+1}(n-1)!\; a^{-n}\sin^n\phi\,\sin n\phi$ Ans.

(b). If $y = \left[x + \sqrt{1+x^2}\right]^m$ then prove that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$

Ans:- On differentiating we get,

$y_1 = m\left[x + \sqrt{1+x^2}\right]^{m-1}\cdot\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$

$\Rightarrow y_1 = \frac{m\left[x+\sqrt{1+x^2}\right]^m}{\sqrt{1+x^2}}$

$\Rightarrow y_1 = \frac{my}{\sqrt{1+x^2}} \Rightarrow (\sqrt{1+x^2})\, y_1 = my$.

Squaring on both sides, we get

$(1+x^2)y_1^2 = m^2y^2$.

Again differentiating w.r. to x, we get

$$2xy_1^2 + (1+x^2)\cdot 2y_1y_2 = 2m^2yy_1$$

$$\Rightarrow \cancel{2}xy_1 + (1+x^2)y_2 = m^2y$$

$$\Rightarrow (1+x^2)y_2 + xy_1 - m^2y = 0.$$

Now, using Leibnitz's theorem, we get

$$(1+x^2)y_{n+2} + {}^nC_1\, y_{n+1}\,(2x) + {}^nC_2\, y_n\cdot(2) + xy_{n+1} + {}^nC_1 y_n - m^2y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

proved.

$\boxed{3}$ (a). If $y = e^{\tan^{-1}x}$, prove that

$(1+x^2)y_{n+2} + [2(n+1)x - 1]\, y_{n+1} + n(n+1)y_n = 0$.

Ans:- Given $y = e^{\tan^{-1}x}$.

differentiating w.r.to x, we get,

$$\frac{dy}{dx} = e^{\tan^{-1}x}\cdot\frac{1}{1+x^2} = \frac{y}{1+x^2}.$$

$$\Rightarrow y_1 = \frac{y}{1+x^2} \Rightarrow (1+x^2)y_1 = y.$$

Again differentiating w.r. to x, we get

$$(1+x^2)y_2 + 2xy_1 = y_1$$

Now, using Leibnitz's theorem, we get

$$(1+x^2)y_{n+2} + {}^nC_1\, y_{n+1}\,(2x) + {}^nC_2\, y_n\,(2) + 2xy_{n+1} + {}^nC_1 y_n\,(2) = y_{n+1}$$

$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + 2xy_{n+1} + 2ny_n - y_{n+1} = 0$

$\Rightarrow (1+x^2)y_{n+2} + [2(n+1)x - 1]\,y_{n+1} + n(n+1)y_n = 0.$

Proved.

$\boxed{3}$ (b) If $y = \cos x \cos 2x . \cos 3x$ find y_n.

Ans:- $y = \cos x \cos 2x . \cos 3x.$

$= \left(\frac{\cos 3x + \cos x}{2}\right) . \cos 3x.$

$= \frac{\cos^2 3x}{2} + \frac{\cos 3x . \cos x}{2} = \frac{\cos 6x + 1}{4} + \frac{\cos 4x + \cos 2x}{4}$

$= \frac{1}{4} + \frac{1}{4}(\cos 2x + \cos 4x + \cos 6x)$

$y_n = D^n\left(\frac{1}{4}\right) + \frac{1}{4} D^n(\cos 2x + \cos 4x + \cos 6x)$

$= 0 + \frac{1}{4}\left[2^n \cos(2x + \frac{n\pi}{2}) + 4^n \cos(4x + \frac{n\pi}{2}) + 6^n \cos(6x + \frac{n\pi}{2})\right]$

$\Rightarrow y_n = \frac{1}{4}\left[2^n \cos(2x + \frac{n\pi}{2}) + 4^n \cos(4x + \frac{n\pi}{2}) + 6^n \cos(6x + \frac{n\pi}{2})\right].$

Ans.

$\left[\because D^n \cos(ax+b) = a^n \cos(ax + b + \frac{n\pi}{2})\right].$

$\boxed{4}$ (a) If $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2 \\ 3 & x = 2. \end{cases}$

Discuss the continuity of $f(x)$ at $x = 2$.

Ans:- Left limit

$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{(2-h)^3-8}{(2-h)^2-4} = \lim_{h\to 0} \frac{8-h^3-12h+6h^2-8}{4-4h+h^2-4}$

$= \lim_{h\to 0} \frac{-h^3+6h^2-12h}{h^2-4h} = \lim_{h\to 0} \frac{-h^2+6h-12}{h-4} = \frac{-12}{-4} = 3.$

Right limit

$\lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{(2+h)^3-8}{(2+h)^2-4} = \lim_{h\to 0} \frac{8+h^3+12h+6h^2-8}{4+4h+h^2-4}$

$= \lim_{h\to 0} \frac{h^3+6h^2+12}{h^2+4h} = \lim_{h\to 0} \frac{h^2+6h+12}{h+4} = \frac{12}{4}$

$= 3.$

As

$$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} f(2+h) = f(2) = 3.$$

So the given function is continuous at $x=2$. 4

(b). Show that

$$\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}\cdots\right).$$

Aw - From Taylor's series expansion, we have.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

Here $f(\theta) = \sin\theta$, $a \Rightarrow \pi/4$, $h \Rightarrow \theta$ $\Rightarrow f(a) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(\theta) = \cos\theta.$ $\Rightarrow f'(\pi/4) = \cos\pi/4 = \frac{1}{\sqrt{2}}$.

$f''(\theta) = -\sin\theta$ $\Rightarrow f''(\pi/4) = -\sin(\pi/4) = \frac{-1}{\sqrt{2}}$

Similarly $f^n(\theta) = \sin(\theta + \frac{n\pi}{2})$ $\Rightarrow f^n(\pi/4) = \sin(\pi/4 + \frac{n\pi}{2})$.

for example $f'''(\theta) = -\cos\theta$ $\Rightarrow f'''(\pi/4) = \frac{-1}{\sqrt{2}}$.

$f^{IV}(\theta) = \sin\theta$ $\Rightarrow f^{IV}(\pi/4) = \frac{1}{\sqrt{2}}$.

So.
$$\sin(\pi/4+\theta) = \sin\left(\frac{\pi}{4}\right) + \theta\cdot\cos\left(\frac{\pi}{4}\right) + \frac{\theta^2}{2!}\left(-\sin\frac{\pi}{4}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \theta\cdot\frac{1}{\sqrt{2}} + \frac{\theta^2}{2!}\cdot\frac{-1}{\sqrt{2}} + \frac{\theta^3}{2!}\cdot\left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{\sqrt{2}}\cdots\right]$$

proved.

$\boxed{5}$ (a) Show that $f(x) = \begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0 & x=0 \end{cases}$

is continuous at $x=0$.

Ans - As $|\sin(\frac{1}{x})| \leq 1$, so $|x \sin(\frac{1}{x})| \leq |x|$.

Hence $\lim_{h\to 0} x \sin(\frac{1}{x}) = 0$.

So $\lim_{h\to 0} x \sin(\frac{1}{x}) = \sin 0 = 0$.

Hence, the given function is continuous at $x = 0$.

(b) State Maclarin's theorem with Lagrange's form of remainder.

Ans:- If $f(x)$ possesses differential coefficients of the first n orders, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x).$$

where θ lies between 0 and 1.

6 (a) Evaluate $\lim_{x\to 0} \frac{\sin x}{\sqrt{x}}$.

Ans - $\lim_{x\to 0} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \sqrt{x}\right) = \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0} \sqrt{x}$

$= 1 \cdot 0 = 0$.

(b) Show that $f(x) = x^2$ is differentiable at $x = 0$ and $x = 1$.

Ans - Given $f(x) = x^2$.

At $x = 0$

Left derivative : $\lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^2 - 0}{-h} = \lim_{h\to 0} -h = 0$

Right derivative : $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 - 0}{h} = \lim_{h\to 0} h = 0$

At $x = 1$

Left derivate: $\lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{(1-h)^2 - 1}{-h} = \lim_{h\to 0} \frac{1-2h+h^2-1}{-h}$

$= \lim_{h\to 0} \frac{h^2 - 2h}{-h} = \lim_{h\to 0} -h+2 = 2$

Similarly Right derivative $= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h}$

$= \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2+h = 2.$

Hence at $x=0$ and $x=1$, left and right derivatives are same respectively. So $f(x)=x^2$ is differentiable at $x=0$ and $x=1$.

7. (a). State and prove Rolle's Theorem.

Ans. Statement

If a function $f: [a,b] \to \mathbb{R}$ is

(i) continuous on the closed interval $[a,b]$,

(ii) derivable in the open interval (a,b), and

(iii) $f(a)=f(b)$, then there exist atleast one point $c \in (a,b)$ such that $f'(c)=0$.

proof:- Since f is continuous on $[a,b]$, it is bounded and attains its supremum and infimum at some point of $[a,b]$.

Let $M =$ supremum of f in $[a,b]$

$m =$ infimum of f in $[a,b]$.

Now, either $M=m$ or $M \neq m$.

If $M=m$, then $f(x)$ is a constant on $[a,b]$

Hence $f'(x)=0 \quad \forall\, x \in [a,b]$.

Thus the theorem holds in this case.

On the other hand if $M \neq m$, then atleast one of M and m, if not both, must different from the equal values $f(a)$ and $f(b)$.

suppose $M \neq f(a) = f(b)$.

Since f attains its supremum in $[a, b]$, there exists $c \in (a, b)$ such that $f(c) = M$.

Also $f'(c)$ exists because of condition (ii).

We claim that $f'(c) = 0$.

If $f'(c) > 0$, then $\exists\ \delta > 0$ such that

$f(x) > f(c) = M \quad \forall\ x \in (c, c+\delta)$.

But $f(x) \leq M \quad \forall\ x \in [a, b]$. $\{\because M$ is the supremum$\}$.

Thus we arrive at a contradiction.

If $f'(c) < 0$, then $\exists\ \delta > 0$ such that

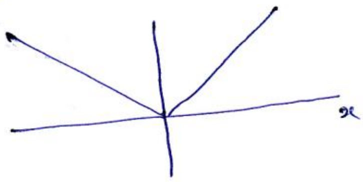
$f(x) > f(c) = M \quad \forall\ x \in (c-\delta, c)$.

which is again not possible.

So $f'(c) = 0$. proved.

(b) Prove that $f(x) = |x|\ \forall\ x \in \mathbb{R}$ is not differentiable at $x = 0$.

Ans Given function is $f(x) = |x| \quad \forall\ x \in \mathbb{R}$



At $x = 0$

Left derivative:

$$\lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Right derivative.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

As left derivative is not equal to right derivative at $x = 0$. So $f(x) = |x|$ is not differentiable at $x = 0$.

proved.

[8]. Sketch the curve.

$$y^2(2a-x) = x^3.$$

Ans (i) The power of y is even. So there is symmetry about x-axis.

(ii) The curve passes through origin.

Eqⁿ of tangents at origin are.

$y^2 = 0$ (on equating the lowest degree term equal to zero).

i.e two coincident tangents. Hence origin is a cusp.

(iii) Except origin the curve does not cross the axes.

(iv) Asymptote parallel to y-axis is

$2a - x = 0$ i.e $x = 2a$ (on equating the coefficient of heighest power of y to zero).

The remaining asymptotes are imaginary.

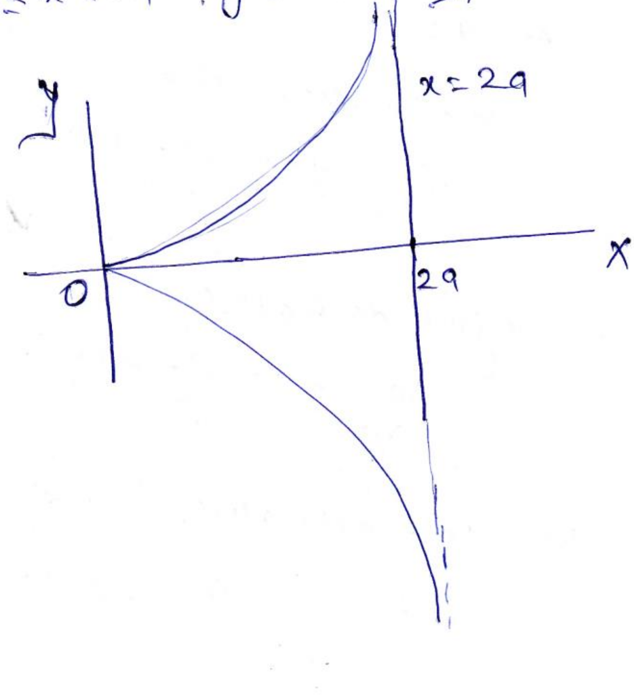
(v) Solving for y, we get

$y = \pm \dfrac{x^{3/2}}{\sqrt{2a-x}}$, so when $x > 2a$; y is imaginary as for $x < 0$ again y is imaginary

Hence curve does not lie on the negative side of x-axis and beyond $x = 2a$.

i.e only for $0 \le x < 2a$, y is real.

The curve is



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